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Metrics compatible with a symmetric connection in dimension three ★

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Abstract

The problem of characterizing symmetric connections that are Levi-Civita connections of a pseudo-Riemannian metric is considered. A more or less complete solution of the problem in three dimensions is presented. In particular the single class of Levi-Civita connections whose metric is not determined up to a conformal factor by the curvature tensor alone is characterized in geometric terms.

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1. Introduction

One of the oldest and apparently most difficult problems of classical differential geometry is to find necessary and sufficient conditions for a given symmetric connection ∇ on an n-dimensional manifold M^n to be the Levi-Civita connection of some metric g. The solution depends on the value of n, but only the case where n is two is truly easy – see Theorem 2.1 and also [12]. It is, however, possible to give an algorithm, that we consider in Section 2, that provides a solution to the problem, as was already known to several classical authors – see for example [7]. This algorithm is only a prescriptive solution and it is the principal aim of the present paper to formulate an analog of Theorem 2.1 in dimension three. Thus an optimal solution would be one that specifies geometric conditions in terms of ∇, the curvature tensor R of ∇ and the higher-order covariant derivatives of R. We propose to be a little vague about the exact smoothness conditions satisfied by ∇. We shall have to suppose

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that various systems of linear equations have solutions of a constant dimension. In practice this can be achieved by restricting to certain open subsets of M^n or else by assuming that ∇ is real analytic. In the smooth category, however, such regularity conditions may not be satisfied. For a further discussion of this point see [11].

The existence problem for metrics has also been considered by Levine [7] in dimension two and in two, three and even n-dimensions by Cheng and Ni [2–4]. Superficially, the approach of the latter authors would appear to resemble the one adopted here. They too consider a system of linear equations similar to that comprised in Eqs. (2.3) and (2.4). This system is solved and the result substituted back into the differential equations that express the compatibility of the metric with the connection. Unfortunately, the very complicated conditions derived, involving polynominal conditions in R and its first and second covariant derivatives, are not written in any tensorial manner so that their geometric significance is far from clear. In addition a certain genericity assumption concerning the solution of the linear system is made, whose meaning for ∇ is uncertain.

In two recent papers Edgar investigates a certain system of homogeneous linear equations [5,6]. (Anticipating developments in Section 2, this system is essentially the same as Eq. (2.8) with k having the value zero.) Edgar is able to obtain some metric existence results by using Petrov's classification of space–time metrics to ensure that this linear system has maximal rank.

The main conclusion of the present paper may be summarized in the following terms. Referring to condition (2.3), arising as integrability conditions from (2.1), which expresses the compatibility of a metric g with ∇, one might hope that g would be determined up to a function multiple if indeed such a g exists. In many cases this is true but in one respect it is false. This exceptional case is covered by Theorem 5.1 with its manifold hypotheses and conditions. The remarks of this last paragraph apply specifically to the $n = 3$ case.

A few more remarks are needed to put Theorem 5.1 – the main result of the paper – into context. The holonomy groups and their Lie algebras for metrics in general and three-dimensional metrics in particular are considered in Besse [1]. All the Lie subalgebras of o(3) and o(2, 1) are known and each is the Lie algebra of the holonomy group of some metric. We shall need to quote the following facts. First of all, the Lie algebra is trivial if and only if the metric is flat. Secondly, the Lie algebra is one-dimensional if and only if either (M^3, g) factors locally as a product of a non-flat two-dimensional metric and (flat) one-dimensional metric or else g is Lorentzian and possesses a parallel null vector field. See also [8,13,14]. In these respective cases coordinates (t, x, z) may be introduced relative to which the metric g assumes the forms:

$$g = c(t)(\mathrm{d}t)^2 + a(x,z)(\mathrm{d}x)^2 + 2h(x,z)\,\mathrm{d}x\,\mathrm{d}z + b(x,z)(\mathrm{d}z)^2, \tag{1.1}$$

$$g = 2\,\mathrm{d}t\,\mathrm{d}z + a(x,z)(\mathrm{d}x)^2 + 2h(x,z)\,\mathrm{d}x\,\mathrm{d}z + b(x,z)(\mathrm{d}z)^2, \tag{1.2}$$

where in (1.1) $c(t)$ is nowhere vanishing as is $ab - h^2$ in both (1.1) and (1.2). Thirdly, the only case in which the Lie algebra is two-dimensional is when g admits a local parallel line distribution: this is equivalent to the existence of a recurrent vector field, say Δ, that cannot be scaled even locally so as to obtain a parallel vector field. In this case the metric may be written locally as in (1.2) except now that b will also depend on t.

Recall that a tensor field T of any type is said to be *recurrent* if the tensor field ∇T satisfies

$$\nabla T = \theta \otimes T \tag{1.3}$$

for some one-form θ. If T is Δ then (1.3) implies that

$$R(X, Y)\Delta = \mathrm{d}\theta(X, Y)\Delta \tag{1.4}$$

for all vector fields X and Y. One sees that a recurrent Δ may be scaled locally to give a parallel vector field if and only if θ is closed.

In practical terms, the Lie algebra of the holonomy group is computed from the curvature and its covariant derivatives. To be specific one considers the endomorphisms $R(X, Y)$, $\nabla_Z R(X, Y), \nabla_W \nabla_Z R(X, Y), \ldots,$ for arbitrary vector fields W, X, Y and Z. All of these endomorphisms are skew-adjoint relative to the metric g if ∇ is Levi-Civita as is the commutator of any two such endomorphisms. Actually one has to exercise some caution here. At any point of M the preceding collection of endomorphisms will generate a Lie algebra which, if ∇ is Levi-Civita, will be a subalgebra of the Lie algebra of the associated orthogonal group. The unique connected Lie group determined by the algebra of endomorphisms is called the infinitesimal holonomy group and is a subgroup of the full holonomy group. In this paper it will be assumed that M is connected and simply connected and that the infinitesimal holonomy group has constant dimension over M and hence coincides with the full holonomy group. For further discussion of these points see [11].

In Theorem 5.1 we shall consider linear subspaces of such endomorphisms. The subspaces generated by R, the first covariant derivatives of R and the second covariant derivatives of R are denoted simply by R, ∇R and $\nabla^2 R$, respectively, in Theorem 5.1. These latter spaces consist simply of linear spans not commutators.

An outline of the paper is as follows. In Section 2 we reconsider the classical prescriptive solution to the problem. In Section 3 we specialize to the three-dimensional case and argue that there are two subcases according to whether the Ricci tensor K has rank one or two. Sections 4–6 are devoted to the rank$(K) = 1$ situation and Section 7 to rank$(K) = 2$. Theorem 5.1 characterizes the exceptional metrics which are determined by second derivatives of R. The main result of Section 7 asserts that the direct product examples are the only ones when rank$(K) = 2$.

It should be mentioned finally that considerable effort has been devoted to the study of the metric existence problem in the physically interesting case of dimension four. The three-dimensional case, however, is sufficiently complicated to make it worthwhile to discuss it separately. The four-dimensional case will be revisited in a future article where some of the points raised by Edgar [5,6] will also be addressed.

2. A prescriptive solution to the problem

Let ∇ be a symmetric connection on an n-dimensional manifold M. Later on, where n plays a role, we shall write M^n instead of M and the majority of the paper is concerned with the case where n is three. We are looking for a metric g that is compatible with ∇ and we regard the compatibility conditions

$$\nabla g = 0 \tag{2.1}$$

as a system of partial differential equations with unknowns the components of g. By "differentiating and equating mixed partials", one obtains the following well-known integrability conditions for (2.1):

$$g(Z,\ )R(X,Y)W + g(W,\ )R(X,Y)Z = 0, \tag{2.2}$$

where W, X, Y and Z are arbitrary vector fields on M. In (2.2) $g(Z,\ )$ for example, is a one-form applied to the vector field which follows it.

Let us abbreviate (2.2) simply to

$$g \circ R + (g \circ R)^{\mathrm{t}} = 0. \tag{2.3}$$

By covariantly differentiating (2.3) repeatedly we obtain a sequence of linear algebraic conditions that we write

$$\begin{aligned} &g \circ \nabla R + (g \circ \nabla R)^{\mathrm{t}} = 0,\\ &g \circ \nabla^2 R + (g \circ \nabla^2 R)^{\mathrm{t}} = 0,\\ &\vdots\\ &g \circ \nabla^k R + (g \circ \nabla^k R)^{\mathrm{t}} = 0. \end{aligned} \tag{2.4}$$

We can now reproduce the classical *prescriptive* solution to the general problem in n-dimensions. A similar discussion may be found in various classical sources – see for example Eisenhart [7]. According to our regularity hypothesis, (2.3) and (2.4) constitute a homogeneous linear system which, at some stage must stabilize; that is, either the system contains sufficiently many conditions that zero is the only solution or else for some value of k, there is a basis of non-zero solutions that remain solutions at the $(k+1)$st stage and hence all higher-order stages. In the former case ∇ admits no compatible metric. In the latter case ∇ will admit a metric if and only if in the span of the basic solutions there is a *non-degenerate* solution $\bar{g}$.

Now $\bar{g}$ is not yet necessarily a metric compatible with g. To construct such a metric, and indeed the most general such metric, weight a basis of solutions by function coefficients chosen so that the resulting linear combination is parallel. This construction yields a first-order system of partial differential equations, which is in fact, a completely integrable total differential system. Thus having once solved the algebraic system (2.3) and (2.4) one obtains a multi-parameter family of metrics compatible with ∇.

The preceding "procedure" does not give a solution in closed form. About all that can be said is that since g has $\binom{n+1}{2}$ components, in (2.3)–(2.4) it is necessary to differentiate at most $\binom{n+1}{2}$ times. A better solution would be one that answers the metric existence problem purely in terms of properties of ∇, R and the higher-order covaraint derivatives of R. Such an optimal situation exists in the case when n is two [12].

Theorem 2.1. *Local necessary and sufficient conditions for a non-flat, symmetric connection ∇ on M^2 to be the Levi-Civita connection of a metric g are that the Ricci tensor of ∇ should be:*
(i) *symmetric and non-degenerate,*
(ii) *recurrent.*
Furthermore if M^2 is simply-connected, g exists globally. □

Let us return to the previous discussion and in arbitrary dimension consider an important special case.

Theorem 2.2. *Suppose that for M^n the solution space to* (2.3)–(2.4) *consists of multiples of a non-degenerate form $\bar{g}$. Then $\bar{g}$ is recurrent and there exists a function λ such that $\lambda\bar{g}$ is parallel.*

Proof. Suppose that system (2.3)–(2.4) "stabilizes" at stage k so that the $(k+1)$st-order covariant derivative conditions are linear combinations of lower-order conditions. Then we have

$$\bar{g}\circ\nabla^k R + (\bar{g}\circ\nabla^k R)^{\mathrm{t}} = 0. \tag{2.5}$$

Covariantly differentiating (2.5) we obtain

$$\nabla\bar{g}\circ\nabla^k R + (\nabla\bar{g}\circ\nabla^k R)^{\mathrm{t}} + \bar{g}\circ\nabla^{k+1} R + (\bar{g}\circ\nabla^{k+1} R)^{\mathrm{t}} = 0. \tag{2.6}$$

But we have identically

$$\bar{g}\circ\nabla^{k+1} R + (\bar{g}\circ\nabla^{k+1} R)^{\mathrm{t}} = 0 \tag{2.7}$$

and hence

$$\nabla\bar{g}\circ\nabla^k R + (\nabla\bar{g}\circ\nabla^k R)^{\mathrm{t}} = 0. \tag{2.8}$$

Since the solution space to (2.3)–(2.4) is spanned by $\bar{g}$, we can only have

$$\nabla\bar{g} = \theta\otimes\bar{g} \tag{2.9}$$

for some one-form θ. Thus $\bar{g}$ is recurrent.

Now if X, Y denote arbitrary vector fields (2.9) gives

$$\nabla_X\nabla_Y\bar{g} - \nabla_Y\nabla_X\bar{g} - \nabla_{[X,Y]}\bar{g} = \mathrm{d}\theta(X,Y)\bar{g}. \tag{2.10}$$

On the other hand, Ricci's identities applied to the left-hand side of (2.10) give

$$\begin{aligned}&\left[\nabla_X\nabla_Y\bar{g} - \nabla_Y\nabla_X\bar{g} - \nabla_{[X,Y]}\bar{g}\right](Z,W)\\ &\quad = -(\bar{g}(W, R(X,Y)Z) + \bar{g}(Z, R(X,Y)W)),\end{aligned} \tag{2.11}$$

where Z and W are further arbitrary vector fields. But $\bar{g}$ satisfies (2.3) so the right-hand side of (2.11) is zero. Thus by (2.10), θ is closed and hence $\bar{g}$ can be scaled by λ so as to obtain a parallel form. □

An important practical consequence of Theorem 2.2 is that if one is seeking a metric compatible with ∇ and in solving (2.3)–(2.4) one reduces to a one-dimensional space spanned by $\bar{g}$, one simply checks whether $\bar{g}$ is recurrent. Clearly this is much easier than constructing $\nabla^{k+1}R$ and then verifying the skew-adjointness conditions.

3. The three-dimensional case

Let us now consider the three-dimensional case. Then (2.3) implies that R has trace zero and if R is not zero, that it has rank two. (More precisely, for arbitrary vector fields X and Y, each of the endomorphisms $R(X, Y)$ has trace zero and rank two.) Considering the possible non-zero Jordan normal forms for R, one sees that the solution space to (2.4) is 0, 1 or 2-dimensional, respectively. In case it is 0, no metric exists and if it is 1, the existence question is answered by Theorem 2.2.

It remains to consider the case where (2.4) has a two-dimensional solution space. In each of the three subcases corresponding to the Jordan normal form of R, this space possesses a non-degenerate form. Furthermore, the curvature matrices must span a one-dimensional space and since they are of rank 0 or 2, they must possess a common kernel spanned by a vector field that we denote by Δ. Thus for arbitrary vector fields X and Y, R satisfies

$$R(X, Y)\Delta = 0. \tag{3.1}$$

If the connection ∇ is to be metric it follows from (3.1) also that

$$R(\Delta, X)Y = 0. \tag{3.2}$$

Condition (3.1) in addition implies that $R(X, Y)Z$ is orthogonal to Δ for all vector fields X, Y and Z. Furthermore, if K denotes the Ricci tensor of ∇ then (3.1) entails that

$$K(X, \Delta) = 0 \tag{3.3}$$

for arbitrary vector fields X. Of course, K must also be symmetric if ∇ is to be metric.

The following lemmas inform us about the solutions to (2.4).

Lemma 3.1. *If the solution space to* (2.4) *is two-dimensional and* ∇ *is to be metric then* K *must be a solution to* (2.4).

Proof. Choose a moving frame (E_i) with E_1 given by Δ. Then in components, K assumes the form

$$K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & R^3_{223} & -R^2_{223} \\ 0 & -R^2_{223} & -R^2_{323} \end{bmatrix}. \tag{3.4}$$

The result now follows from (3.1) and (3.2). □

Next suppose that a metric g indeed has been found that is compatible with ∇. Denote the one-form dual to Δ via g by α.

Lemma 3.2. *The symmetric square of* α *satisfies* (2.4).

Proof. Let W, X, Y and Z be arbitrary vector fields. Then

$$\alpha(X)\alpha(Y) = g(\Delta, X)g(\Delta, Y). \tag{3.5}$$

Denote $\alpha(X)\alpha(Y)$ by $h(X, Y)$. Then

$$\begin{aligned}(h \circ R)(X, Y)(Z, W) &= g(\Delta, R(X, Y)Z)g(\Delta, W)\\ &= -g(Z, R(X, Y)\Delta)g(\Delta, W)\\ &= 0.\end{aligned}$$

Thus h satisfies (2.4) a fortiori. □

The discussion now divides into two parts according as the rank of K is one or two.

4. Rank of $K = 1$: Necessary conditions

In the case where the rank of K is one, Lemmas 3.1 and 3.2 imply that there exists some nowhere vanishing function λ such that for arbitrary vector fields X and Y

$$K(X, Y) = \lambda g(\Delta, X)g(\Delta, Y). \tag{4.1}$$

Clearly then (3.3) entails that Δ is null and hence also that $\nabla_X \Delta \in D$ for all vector fields X, where D denotes the distribution orthogonal to Δ. Furthermore, the scalar curvature of g must be zero.

We shall establish two facts about the distribution D.

Proposition 4.1. *The degenerate distribution of* K *coincides with* D.

Proof. Let Y be an arbitrary vector field. Then X belongs to the degenerate distribution of K iff

$$\lambda g(\Delta, X)g(\Delta, Y) = 0 \tag{4.2}$$

in view of (3.5). Since Y is arbitrary and λ non-zero, this is in turn equivalent to

$$g(\Delta, X) = 0.$$ □

Proposition 4.2. *The distribution D is integrable if and only if Δ is a pre-geodesic, that is, $\nabla_\Delta \Delta$ is a multiple of Δ.*

Proof. Suppose that X is tangent to D, that is,

$$g(\Delta, X) = 0. \tag{4.3}$$

Covariantly differentiating along Δ one obtains

$$g(\nabla_\Delta X, \Delta) + g(X, \nabla_\Delta \Delta) = 0. \tag{4.4}$$

Next, using the fact that Δ is null and that ∇ is symmetric we obtain

$$g(X, \nabla_\Delta \Delta) + g(\Delta, [\Delta, X]) = 0. \tag{4.5}$$

Now if D is integrable the second term above is zero and hence $\nabla_\Delta \Delta$ is orthogonal to D, that is, proportional to Δ. The converse is apparent from (4.5). □

Note also that from (3.3), (3.5) and the fact that Δ is null we find that $K(\nabla_X \Delta, Y)$ vanishes for all vector fields X and Y.

Let us recall the following formula valid for three-dimensional metric manifolds X, Y and Z being arbitrary vector fields, r the scalar curvature and G the metric inverse to g:

$$\begin{aligned} R(X,Y)Z = {} & K(X,Z)Y - K(Y,Z)X + g(X,Z)(G \circ K(Y)) \\ & - g(Y,Z)(G \circ K(X)) - \tfrac{1}{2} r (g(X,Z)Y - g(Y,Z)X). \end{aligned} \tag{4.6}$$

As we stated at the start of this section, r must be zero hence from (4.1) we obtain

$$\begin{aligned} R(X,Y)Z = \lambda\{ & g(\Delta, X)(g(\Delta, Z)Y - g(Y,Z)\Delta) \\ & - g(\Delta, Y)(g(\Delta, Z)X - g(X,Z)\Delta)\}. \end{aligned} \tag{4.7}$$

Introducing a fourth vector field W, we find from (4.7) that the covariant derivative $\nabla_W R$ is given by

$$\begin{aligned} (\nabla_W R)(X,Y)Z = {} & (W\lambda/\lambda) R(X,Y)Z \\ & + \lambda\{(g(\Delta, Y)g(X,Z) - g(\Delta, X)g(Y,Z))\nabla_W \Delta \\ & + (g(\nabla_W \Delta, Y)g(X,Z) - g(\nabla_W \Delta, X)g(Y,Z))\Delta \\ & - (g(\nabla_W \Delta, Y)g(\Delta, Z) + g(\Delta, Y)g(\nabla_W \Delta, Z))X \\ & + (g(\nabla_W \Delta, X)g(\Delta, Z) + g(\Delta, X)g(\nabla_W \Delta, Z))Y\}. \end{aligned} \tag{4.8}$$

One consequence of (4.8) is that

$$\nabla_\Delta R(X,Y)Z = (\Delta\lambda/\lambda) R(X,Y)Z. \tag{4.9}$$

From (4.8) one may construct the second Bianchi identity for ∇. On replacing Z by Δ and making use of the identity

$$\mathrm{d}\alpha(X,Y) = (\nabla_X\alpha)(Y) - (\nabla_Y\alpha)(X), \tag{4.10}$$

one obtains the condition

$$(\alpha \wedge \mathrm{d}\alpha)(W,X,Y) = 0, \tag{4.11}$$

where α is the one-form dual to Δ via g. Thus the one-form α and the distribution D are integrable. In addition by Proposition 4.2, Δ is a pre-geodesic and hence may be scaled locally so as to obtain a bona fide geodesic field.

We continue by deriving further necessary conditions for the existence of a metric g compatible with ∇.

Proposition 4.3. *Let W, X, Y be arbitrary vector fields and $Z \in D$. Then $\nabla_W R(X,Y)Z$ is a multiple of Δ.*

Proof. First of all note that in light of (4.6) and (4.7) we have

$$\begin{aligned}\nabla_W R(X,Y)Z \equiv \lambda\{&(g(\Delta,Y)g(X,Z) - g(\Delta,X)g(Y,Z))\nabla_W\Delta \\ &+g(\nabla_W\Delta,Z)(g(\Delta,X)Y - g(\Delta,Y)X)\},\end{aligned} \tag{4.12}$$

where the congruence denotes working modulo multiples of Δ. Now consider

$$\begin{aligned}&g(Z,\nabla_W R(X,Y)Z) \\ &\quad= \lambda\{((g(\Delta,Y)g(X,Z) - g(\Delta,X)g(Y,Z))g(Z,\nabla_W\Delta)) \\ &\qquad+g(\nabla_W\Delta,Z)(g(\Delta,X)g(Y,Z) - g(\Delta,Y)g(X,Z))\} = 0.\end{aligned} \tag{4.13}$$

Since D is two-dimensional and contains Δ it follows that $\nabla_W R(X,Y)Z$ is a multiple of Δ. □

We have seen that the curvature matrices alone span a one-dimensional space. In view of Proposition 4.3 and the assumption that K is symmetric, it follows that the curvature and first covariant derivatives span precisely a two-dimensional space. Indeed if it were one-dimensional the infinitesimal holonomy group too would be one-dimensional and, locally at least, a metric g compatible with ∇ would possess a parallel, null vector field. The fact that R and its first covariant derivatives span *at most* a two-dimensional space bearing in mind that any solution to (2.4) must be a trace-free matrix of even rank, follows from Proposition 4.3.

Let us take note of the following identities that follow from (3.2), (4,7), the fact that Δ is pre-geodesic and the use of Ricci's identities to commute the order of covariant differentiation, V, W, X, Y and Z denoting arbitrary vector fields:

$$\nabla_\Delta R(X,Y)Z = (\Delta\lambda/\lambda)R(X,Y)Z, \tag{4.14}$$

$$\nabla_W R(\Delta,Y)Z = -R(\nabla_W\Delta,Y)Z, \tag{4.15}$$

$$\nabla_\Delta \nabla_W R(X,Y)Z = W(\Delta\lambda/\lambda)R(X,Y)Z + (\Delta\lambda/\lambda)\nabla_W R(X,Y)Z, \tag{4.16}$$

$$\nabla_V \nabla_W R(\Delta,Y)Z = -\nabla_V(R(\nabla_W \Delta, Y)Z), \tag{4.17}$$

$$\nabla_V \nabla_W R(X,Y)\Delta \in D. \tag{4.18}$$

Next recall from Section 2 that the infinitesimal holonomy group in view of (3.1) cannot be two-dimensional. Since it can be at most three-dimensional it must be precisely three-dimensional and must be spanned by R and its first- and second-covariant derivatives.

We derive one further necessary condition for the existence of g. To this end, note that $\nabla_V \nabla_W R(X,Y) - \nabla_{\nabla_W V} R(X,Y)$, where V, W, X and Y are arbitrary vector fields, must be skew-adjoint relative to any putative metric g compatible with ∇. It follows that this quantity, interpreted as an endomorphism field, must have even rank and in particular must be singular. (The reason that we consider this quantity rather than $\nabla_V \nabla_W R(X,Y)$ is that this latter quantity is not tensorial in W.)

5. Rank of $K = 1$: Sufficient conditions

In this section we combine several of the necessary conditions derived in the previous section with some others so as to obtain a set of conditions that are necessary and sufficient for the existence of a metric g that is compatible with ∇. Let us state formally the following theorem.

Theorem 5.1. *Let ∇ be a symmetric connection on M^3 whose associated Ricci tensor K has rank one and let the degenerate distribution of K be denoted by D. Assume that ∇ possesses no parallel vector field and that Eq.* (2.4) *does determine a non-degenerate quadratic form which is not unique up to multiples. Then the following conditions are necessary and sufficient for the existence of a metric g that is compatible with ∇ (U, V, W, X and Y denoting arbitrary vector fields*):

1. *There exists a vector field Δ such that $R(X,Y)\Delta = R(\Delta, X)Y = 0$.*
2. *K is symmetric.*
3. *D is integrable.*
4. *$K(\nabla_X \Delta, Y) = 0$.*
5. *$\nabla_\Delta R(X,Y)V$ is a multiple of $R(X,Y)V$.*
6. *$\nabla_W R(X,Y)Z$ is a multiple of Δ where $Z \in D$.*
7. *The endomorphism $\nabla_V \nabla_W R(X,Y) - \nabla_{\nabla_W V} R(X,Y)$ is singular.*
8. *All commutators of R and ∇R as well as all commutators of ∇R and ∇R, lie in the space spanned by R: symbolically $[R, \nabla R] \subset R$ and $[\nabla R, \nabla R] \subset R$.*
9. *All commutators of R and $\nabla^2 R$ lie in the space spanned by R and $\nabla^2 R$: symbolically* $[R, \nabla^2 R] \subset \mathrm{span}(R, \nabla R)$.

Proof. It appears to be necessary to give the proof in terms of local coordinates. We choose a coordinate system (x^i) in which Δ is represented by $\partial/\partial x^1$. Then by condition (1), R^i_{jkl}

vanishes whenever j, k or l takes the value 1. Furthermore, since K is symmetric and has rank one and since D is integrable by (3.4), we may assume that R^2_{2kl} and R^3_{2kl} are all zero.

We now adopt the procedure described in Section 2 to ascertain the existence of a metric g. Indeed the algebraic solution to (2.4) may be written in the form

$$r\begin{bmatrix} 0 & 0 & R^2_{323} \\ 0 & -R^1_{223} & -R^1_{323} \\ R^2_{323} & -R^1_{323} & 0 \end{bmatrix} + s\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5.1}$$

Let us now consider the covariant derivatives of R. In view of (4.15), which follows from conditions (1) and (4) if suffices to consider $R^i_{jkl;m}$ with m taking on the values 2 and 3 and $k = 2$ and $l = 3$. Note that the symmetry of K entails that $R^i_{ikl;m}$ (sum over i) must be zero. Next because of condition (5), $R^1_{j23;m}$ with $m = 2$ or 3 is necessarily of the form

$$\begin{bmatrix} R^1_{123;m} & R^1_{223;m} & R^1_{323;m} \\ 0 & 0 & -R^2_{323;m} \\ 0 & 0 & -R^1_{123;m} \end{bmatrix}. \tag{5.2}$$

On putting $r = 1$ in (5.1) and demanding that (5.1) be skew-adjoint relative to (5.2) we obtain the conditions:

$$R^1_{223}R^2_{323;m} - R^1_{323}R^1_{123;m} - R^2_{323}R^1_{223;m} = 0, \tag{5.3}$$

$$sR^1_{123;m} + R^1_{323}R^2_{323;m} - R^2_{323}R^1_{323;m} = 0. \tag{5.4}$$

An elementary calculation reveals that (5.3) is equivalent to the condition that a commutator of elements of R and ∇R lie in R, that is, $[R, \nabla R] \subset R$. Hence (5.3) is satisfied by virtue of condition (8).

Turning to (5.4), there is just a single curvature condition obtained by setting $m = 2$ then $m = 3$ and eliminating s between the resulting equations. On making use of (5.3) this single condition may be written as

$$R^1_{323}\left[R^2_{323;3}R^1_{223;2} - R^1_{223;3}R^2_{323;2}\right] + R^1_{223}\left[R^1_{323;3}R^2_{323;2} - R^1_{323;2}R^2_{323;3}\right] + R^2_{323}\left[R^1_{223;3}R^1_{323;2} - R^1_{223;2}R^1_{323;3}\right] = 0. \tag{5.5}$$

But (5.5) is easily seen to be precisely the condition that $[\nabla R, \nabla R] \subset R$ and hence (5.5) is satisfied because of condition (8).

Eq. (5.4) now determines our putative metric g up to a multiple. It remains to check that the skew-adjointness conditions are satisfied for the second-order covariant derivatives of R. Notice that by virtue of conditions (4) and (6), Eq. (4.18) holds, or equivalently, $R^3_{ikl;mn}$ vanishes. Also we have the following identity that results from condition (1) by differentiating twice:

$$\nabla_V\nabla_W R(\Delta, Y)U = \nabla_V R(\nabla_W\Delta, Y)U - \nabla_W R(\nabla_V\Delta, Y)U - R(\nabla_V\nabla_W\Delta, Y)U. \tag{5.6}$$

Thus the skew-adjointness condition for $\nabla_V \nabla_W R(\Delta, Y)U$ is a consequence of that for R and ∇R.

Similarly, if we write the function of proportionality implicit in condition (5) in the form $\Delta\lambda/\lambda$ so as to agree with the notation of Section 5, we find on differentiation

$$\nabla_W \nabla_\Delta R(X,Y)U = W\left(\frac{\Delta\lambda}{\lambda}\right) R(X,Y)U + \frac{\Delta\lambda}{\lambda}\nabla_W R(X,Y)U, \tag{5.7}$$

so that the left-hand side of (5.6) satisfies the skew-adjointness conditions. Furthermore, making use of Ricci's identity, namely,

$$\begin{aligned}&\nabla_V \nabla_W R(X,Y)U - \nabla_W \nabla_V R(X,Y)U \\ &\quad = \nabla_{[V,W]} R(X,Y)U + R(V,W)R(X,Y)U - R(R(X,V),Y)U \\ &\qquad -R(X,R(Y,V)W)U - R(X,Y)R(V,W)U,\end{aligned} \tag{5.8}$$

and replacing U by Δ, we see that the skew-adjointness conditions are satisfied by $\nabla_\Delta \nabla_W R(X,Y)U$ as well.

From the preceding considerations we only have to check the skew-adjointness conditions for the matrices $R^i_{j23;kl}$with$(k,l) = (2,2), (k,l) = (2,3)$ and $(k,l) = (3,3)$. These skew-adjointness conditions turn out to be:

$$R^2_{323} R^3_{223;kl} - R^1_{223} R^2_{123;kl} = 0, \tag{5.9}$$

$$R^1_{323} R^2_{223;kl} + R^1_{223} R^2_{223;kl} = 0, \tag{5.10}$$

$$R^1_{323} R^3_{123;kl} + R^2_{323} R^2_{223;kl} = 0, \tag{5.11}$$

$$R^2_{323} R^1_{223;kl} - R^1_{223} R^2_{323;kl} + R^1_{323} R^1_{123;kl} + s R^3_{223;kl} = 0, \tag{5.12}$$

$$R^2_{323} R^1_{323;kl} - R^1_{323} R^2_{323;kl} + s R^3_{323;kl} = 0, \tag{5.13}$$

where s is determined from (5.4).

With regard to (5.9)–(5.11) note that *algebraically* only two independent conditions appear. Moreover, it is easy to check that these conditions are identically satisfied in virtue of (5.2) and the zero components of R itself. (For instance condition (6) implies that

$$\Gamma^2_{ik} R^1_{223} - \Gamma^3_{12} R^2_{323} = 0 \quad (k = 2,3). \tag{5.14}$$

Also one finds that

$$R^3_{223;kl} = -\Gamma^3_{2k}\Gamma^3_{2l} R^2_{323}, \tag{5.15}$$

etc.)

Turning to (5.12), this is equivalent to condition (9). The argument for this is similar to the one that shows that (5.3) and (5.4) are equivalent to condition (9) making use also of (5.14) and (5.15).

Finally, we show that (5.12) is satisfied. Taking into account (5.9)–(5.11), (5.12) may be written in the form

$$R^2_{323}R^1_{223;kl} - R^1_{223}R^2_{323;kl} + \frac{R^1_{323}\left[R^1_{323}R^3_{223;kl} - R^1_{223}R^3_{323;kl}\right]}{R^1_{223}} + sR^3_{223;kl} = 0. \tag{5.16}$$

Now use condition (7), which implies that $R^i_{j23;kl}$ has determinant zero and recall that its trace is also zero. Combining these two facts with (5.15) gives (5.12) as claimed. □

6. Metrics and connections corresponding to Theorem 5.1

Are there indeed connections that satisfy the many hypotheses of Theorem 5.1? We shall answer this question affirmatively beginning with the following result.

Proposition 6.1. *Let Δ be a vector field on a manifold with a metric g and suppose that Δ is geodesic and satisfies $R(X,Y)\Delta = 0$. Then the third Lie derivative of g with respect to Δ is zero.*

Proof. Starting from the identity

$$(L_\Delta g)(X,Y) = g(\nabla_X\Delta, Y) + g(X, \nabla_Y\Delta), \tag{6.1}$$

using $R(X,Y)\Delta = 0$ and the fact that Δ is geodesic we find that

$$(L^2_\Delta g)(X,Y) = 2g(\nabla_X\Delta, \nabla_Y\Delta). \tag{6.2}$$

Differentiating (6.2) and using (6.2) again we find immediately

$$(L^3_\Delta g)(X,Y) = 0. \qquad \square \tag{6.3}$$

To construct the desired metric g, we introduce coordinates (t,x,z) in which Δ is $\partial/\partial t$ and the orthogonal distribution D of Δ, necessarily integrable, is spanned by $\partial/\partial t$ and $\partial/\partial x$. Furthermore, the non-zero components of g may be assumed to be quadratic in t in view of Proposition 6.1. We may also make coordinate changes of the form

$$\bar{t} = \varphi(x,z)t + \theta(x,z), \qquad \bar{x} = \bar{x}(x,z), \qquad \bar{z} = z(x,z), \tag{6.4}$$

which will preserve the restrictions already imposed on g. Accordingly g may be assumed to be of the local form

$$\begin{aligned} g = {} & (At^2 + 2Dt + H)(\mathrm{d}x)^2 + 2(Ct^2 + 2Et + F)\,\mathrm{d}x\,\mathrm{d}z \\ & + (Bt^2 + 2Gt + K)(\mathrm{d}z)^2 + 2\,\mathrm{d}z\,\mathrm{d}t, \end{aligned} \tag{6.5}$$

where A is unity or zero and $B, C, \ldots, K$ are functions of x and z.

After a lengthy calculation involving computing the Levi-Civita connection of g given by (6.5) and the corresponding curvature tensor, one finds that all the conditions of Theorem 5.1 are satisfied if and only if, after an appropriate coordinate transformation, g is given by

$$g = (\mathrm{d}x)^2 + 2Ct^2\,\mathrm{d}x\,\mathrm{d}z + (C^2t^2 - 2C_xt)(\mathrm{d}z)^2 + 2\,\mathrm{d}z\,\mathrm{d}t, \tag{6.6}$$

where C is a non-zero function of x and z. Thus (6.6) and its associated connection provide a local coordinate normal form for a metric and connection of the type occurring in Theorem 5.1.

7. Rank of $K = 2$

In the case where K is of rank two a metric g compatible with ∇ must be of the form

$$g(X,Y) = \frac{2K(X,Y)}{r} + \mu g(\Delta, X)g(\Delta, Y) \tag{7.1}$$

for some nowhere vanishing functions r and μ. By scaling Δ we may assume that μ is ± 1 in (7.1) and correspondingly that Δ has squared length ± 1. By replacing g by $-g$ we may assume further that $g(\Delta, \Delta)$ are μ, are both unity. In this case r represents the scalar curvature of g in (7.1). Note that Δ cannot be null or else g would be degenerate.

Assuming then that μ is unity in (7.1) it follows that

$$G \circ K = \tfrac{1}{2}r(I - \Delta \otimes g(\Delta,-)). \tag{7.2}$$

Substituting (7.2) into (4.6) we obtain

$$\begin{aligned} R(X,Y)Z = {} & K(X,Z)Y - K(Y,Z)X \\ & + \tfrac{1}{2}r(g(\Delta,X)g(Y,Z) - g(\Delta,Y)g(X,Z))\Delta. \end{aligned} \tag{7.3}$$

From (6.3) and with W another field we find

$$\begin{aligned} \nabla_W R(X,Y)Z = {} & (\nabla_W K(X,Z))Y - (\nabla_W K(Y,Z))X \\ & + (Wr/r)(R(X,Y)Z - K(X,Z)Y + K(Y,Z)X) \\ & + \tfrac{1}{2}r(g(\nabla_W \Delta, X)g(Y,Z) - g(\nabla_W \Delta, Y)g(X,Z))\Delta \\ & + \tfrac{1}{2}r(g(\Delta, X)g(Y,Z) - g(\Delta, Y)g(X,Z))\nabla_W \Delta. \end{aligned} \tag{7.4}$$

Again putting $Z = \Delta$ in the second Bianchi identity and using the fact that

$$\nabla_Y K(W,\Delta) - \nabla_W K(Y,\Delta) = \tfrac{1}{2}r(g(Y, \nabla_W \Delta) - g(W, \nabla_Y \Delta)) \tag{7.5}$$

we find that the analog of (4.11) is

$$\mathrm{d}\alpha(Y,W)X + \mathrm{d}\alpha(X,Y)W + \mathrm{d}\alpha(W,X)Y - \alpha \wedge \mathrm{d}\alpha(X,Y,W)\Delta = 0, \tag{7.6}$$

α being the one-form dual to Δ.

In (7.6) X, Y and W are arbitrary. Let us put $W = \Delta$ in (7.6), use (4.10) and also the fact that $g(\Delta, \Delta)$ is unity to obtain

$$\mathrm{d}\alpha(Y,\Delta) = -g(\nabla_\Delta \Delta, Y) \tag{7.7}$$

and hence since X and Y remain arbitrary

$$\nabla_\Delta \Delta = 0. \tag{7.8}$$

Thus Δ is a geodesic field.

Next, rewrite (7.6) in terms of g as

$$\begin{aligned}
&(g(\nabla_Y\Delta, W) - g(Y, \nabla_W\Delta))(X - g(\Delta, X)\Delta) \\
&\quad +(g(\nabla_W\Delta, X) - g(W, \nabla_X\Delta))(Y - g(\Delta, Y)\Delta) \\
&\quad +(g(\nabla_X\Delta, Y) - g(X, \nabla_Y\Delta))(W - g(\Delta, W)\Delta) = 0.
\end{aligned} \tag{7.9}$$

In view of the fact that Δ has length unity and is geodesic (7.9) is in turn equivalent to

$$\begin{aligned}
&(g(\nabla_Y\Delta, W) - g(Y, \nabla_W\Delta))X + (g(\nabla_W\Delta, X) - g(W, \nabla_X\Delta))Y \\
&\quad +(g(\nabla_X\Delta, Y) - g(X, \nabla_Y\Delta))W = 0.
\end{aligned} \tag{7.10}$$

Since W, X and Y are arbitrary in (7.10) it follows that

$$g(\nabla_X\Delta, Y) - g(X, \nabla_Y\Delta) = 0 \tag{7.11}$$

for arbitrary X and Y. Thus the two-dimensional distribution D orthogonal to Δ is integrable.

Consider next the one-form $g(\Delta,\ )$. We find that its Lie derivative along Δ is given by, X denoting an arbitrary vector field,

$$L_\Delta(g(\Delta,\ ))X = g(\nabla_\Delta\Delta, X) + g(\Delta, \nabla_X\Delta) \tag{7.12}$$

and hence $L_\Delta(g(\Delta,\ ))$ is zero. From (7.1) one thus obtains

$$L_\Delta g = \frac{2}{r}\left(L_\Delta K - \frac{\Delta r}{r}K\right). \tag{7.13}$$

On the other hand let us compute the Lie derivative of R from (7.3), namely,

$$\begin{aligned}
(L_\Delta R)(X,Y)Z = {} & (L_\Delta K(X,Z))Y - (L_\Delta K(Y,Z))X \\
& + \frac{\Delta r}{r}(R(X,Y)Z - K(X,Z)Y + K(Y,Z)X).
\end{aligned} \tag{7.14}$$

Taking the trace in (7.14) we obtain

$$L_\Delta K = (\Delta r/r)K. \tag{7.15}$$

From (7.13) and (7.15) it follows that

$$L_\Delta g = 0, \tag{7.16}$$

that is, Δ is necessarily a Killing vector field. In conjunction with (7.11), this latter fact entails that Δ is a parallel vector field and hence, since Δ is non-null, the de Rham theorem implies that (M^3, g) admits a local product decomposition.

In summary we have the following theorem.

Theorem 7.1. *Suppose that ∇ is a symmetric connection on M^3 whose Ricci tensor K has rank two and that Eq.* (2.4) *does determine a non-degenerate quadratic form which is not unique up to multiples. Then if ∇ is to be metric, this metric and hence ∇ must factor locally as a de Rham product of one- and two-dimensional constituents.* □

As a consequence of Eq. (4.1) and Theorem 7.1 we can reach the following conclusion.

Corollary 7.2. *If ∇ is a symmetric connection on M^3 and ∇ is to be the Levi-Civita connection of a Riemannian metric then g is determined by R alone in the sense of Theorem* 2.2. □

We can also obtain the following result easily.

Theorem 7.3. *Suppose that ∇ is a symmetric connection on M^3. Then ∇ is locally the Levi-Civita connection of a de Rham product of a one-dimensional and non-flat two dimensional metric manifolds iff the following conditions are satisfied, W, X, Y, Z denoting arbitrary vector fields:*

(i) $K(X,Y) = K(Y,X)$,

(ii) *rank of* $K = 2$,

(iii) *K is recurrent, that is, there is a one-form θ such that* $\nabla K = \theta \otimes K$,

(iv) $R(X,Y)Z = K(X,Z)Y - K(Y,Z)X$.

Proof. The given conditions are all clearly necessary. Conversely, starting from (iii) one finds that

$$\nabla_X \nabla_Y K - \nabla_Y \nabla_X K - \nabla_{[X,Y]} K = \mathrm{d}\theta(X,Y)K. \tag{7.17}$$

On the other hand, we also have identically that

$$\begin{aligned}(\nabla_X \nabla_Y K - \nabla_Y \nabla_X K - \nabla_{[X,Y]} K)(Z,W) \\ = K(Z,\ ) \bullet R(X,Y)W - K(W,\ )R(X,Y)Z\end{aligned} \tag{7.18}$$

(cf. (5.8)). From conditions (i) and (iv) we find that the left-hand side of (7.18) and hence of (7.17) vanishes. Thus θ is closed and locally we may write for some function f

$$\theta = \mathrm{d}f. \tag{7.19}$$

Now we observe that

$$\nabla(\mathrm{e}^{-f} K) = \mathrm{e}^{-f}(\nabla K - \mathrm{d}f \otimes K) = 0. \tag{7.20}$$

We define a distribution D spanned by the image of all vector fields $R(X,Y)Z$. Note that in view of (iii) and (iv), R is recurrent and hence D is integrable. We also choose a vector field Δ that spans the degenerate distribution of K and note that since K is recurrent so too is Δ. Indeed write

$$\nabla \Delta = \varphi \otimes \Delta \tag{7.21}$$

for some one-form φ. Now by virtue of (iv) we find that

$$R(X,Y)\Delta = 0. \tag{7.22}$$

From (7.21) and (7.22) it follows that φ is closed and hence Δ may be locally rescaled so as to give a parallel vector field that we continue to denote by Δ.

Now define a metric g on M^3 by declaring Δ to have unit length, Δ and D to be orthogonal and on D define g to be $e^{-f}K$. We show that ∇ is compatible with g, hence by the uniqueness aspect of the fundamental lemma of pseudo-Riemannian geometry, ∇ must be the Levi-Civita connection of g.

Clearly, since $e^{-f}K$ is parallel, we need only check that the one-form $g(\Delta,\ )$ is parallel. First of all note that since Δ is parallel,

$$(\nabla_X(g(\Delta,\ )))Y = X(g(\Delta, Y)) - g(\Delta, \nabla_X Y). \tag{7.23}$$

If Y is Δ then clearly the right-hand side of (7.23) is zero. On the other hand, if $Y \in D$ the right-hand side of (7.23) is also zero since Y is orthogonal to Δ and D is a parallel distribution. This is sufficient to show that $g(\Delta,\ )$ is parallel. □

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